UNIT 4 : INFINITE SERIES I

UNIT STRUCTURE

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4.1 LEARNING OBJECTIVES

After going through this unit, you will able to-

- infinite series, convergence and divergence of infinite series
- Cauchy's Principle of Convergence, for series
- Comparison test for convergence
- divergence of series
- p-series.

4.2 INTRODUCTION

The study of infinite series is an indispensable part of analysis. In this unit we will define infinite series, convergence and divergence of infinite series, Cauchy's Principle of convergence for series. We will also discuss comparion test for convergence and divergence of series, p-series.

4.3 INFINITE SERIES

Let us consider the sequence

$$\left\{\frac{1}{n(n+1)}\right\} \text{ i.e. } \left\{\frac{1}{1\times 2}, \frac{1}{2\times 3}, \frac{1}{3\times 4}, \ldots\ldots\ldots\right\}$$

From the terms of the sequence, we get a summation of the infinite number of terms of the sequence, viz,

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots\ldots\ldots\infty$$

Such an expression is called an infinite series.

Thus, we define an infinite series as follows.

If $\{u_n\}$ is a sequence of real numbers, then the expression

$$u_1+u_2+u_3+\ldots\ldots\ldots\infty \qquad (1)$$

is called an infinite series of real numbers, or simply, infinite series or series.

It is, generally, written as $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$

Example : $\sum \frac{1}{n\,(n+1)}$ is an infinite series.

$$\sum \frac{1}{n\,(n+1)}=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots\ldots\ldots\infty$$

From the series (1), we from the sums

S_1 = first partial sum = u_1

S_2 = second partial sum = $u_1 + u_2$

S_3 = third partial sum = $u_1 + u_2 + u_3$

.....

S_n = nth partial sum = $u_1 + u_2 + u_3 + \ldots\ldots\ldots + u_n$

The sequence $\{S_n\}$ is called the sequence of partial sums of the series $\sum u_n$

4.4 CONVERGENCE AND DIVERGENCE OF INFINITE SERIES

Let us consider the series

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + ...\infty$$

If the sequence of partial sums $\{S_n\}$ is convergent, i.e., if there exists a number S such that $\lim_{n \to \infty} S_n = s$ then the series is called convergent, and S is called the sum of the series.

If $\lim_{n \to \infty} S_n$ does not exist, the series is called divergent. In this case the sum of the infinite series does not exist. Thus, we see that all infinite series cannot be summed.

Examples 1 : Let us consider the infinite series 1 + 2 + 3 + ...∞

Here $S_n = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{n}$

clearly $S_n \to \infty$, as $n \to \infty$

$\therefore$ The given series is not convergent. The series cannot of summed.

2. Let us consider the series

$$1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}} + ...\infty$$

Here $S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}}$

$$= \frac{1\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}}, \text{ (geometric series with common ratio } \frac{1}{2}\text{)}$$

$$= 2\left(1 - \frac{1}{2^n}\right)$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} 2\left(1 - \frac{1}{2^n}\right) = 2$$

$\therefore$ the series is convergent, and it converges to 2, i.e., 2 is the sum of the series.

3. Let us consider the series

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + ... + \frac{1}{n(n+1)} + ...\infty$$

Here $u_r = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$

$$\therefore u_1 = 1 - \frac{1}{2}$$

$$u_2 = \frac{1}{2} - \frac{1}{3}$$

$$u_3 = \frac{1}{3} - \frac{1}{4}$$

.....

.....

$$u_{n-1} = \frac{1}{n-1} - \frac{1}{n}$$

$$u_n = \frac{1}{n} - \frac{1}{n+1}$$

$$\therefore S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$$\therefore \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}} = 1$$

i.e. $\sum_{n=1}^{\infty} \frac{n}{n+1} = 1$

Note : $\sum_{n=1}^{\infty} u_n$ denotes not only the infinite series, but also its sum.

Do You Know ?

1. Addition of a finite number of terms to an infinite series does not affect the convergence or divergence.
2. Removal of a finite number of terms from an infinite series does not affect the convergence or divergence.
3. Multiplication of each term of an infinite series by a nonzero constant does not affect the convergence or divergence.

Theorem 4.1 If $\sum u_n$ is convergent, then $\lim_{n \to \infty} u_n = 0$

Proof : We have

$$S_n = u_1 + u_2 + u_3 + ... + u_{n-1} + u_n$$

$$S_{n-1} = u_1 + u_2 + u_3 + ... + u_{n-1}$$

$\therefore u_n = S_n - S_{n-1}$

$\because \sum u_n$ is convergent, there exists a real number s such that

$$\lim_{n \to \infty} S_n = S$$

Also $$\lim_{n \to \infty} S_{n-1} = S$$

Now $$\lim_{n \to \infty} \sum u_n = \lim_{n \to \infty} (S_n - S_{n-1})$$

$$= \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1}$$

$$= S - S = 0$$

Note : 1. We have seen that if $\sum u_n$ is convergent, $\lim_{n \to \infty} u_n = 0$.

Contrapositively, if $\lim_{n \to \infty} u_n \neq 0$, the series is not convergent.

Let us consider the series $\sum_{n=1}^{\infty} \frac{2n^2}{3n^2 - 1}$

Here $u_n = \frac{2n^2}{3n^2 - 1}$

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{2n^2}{3n^2 - 1} = \lim_{n \to \infty} \frac{2n^2}{3 - \frac{1}{n^2}} = \frac{2}{3} \neq 0$$

$\therefore$ The series is not convergent.

2. The converse of Theorem 4.1 is not true. If , $\lim_{n \to \infty} u_n = 0$ the series may or may not converge.

Theorem 4.2 Let the series $\sum u_n$ converge to s, and the series $\sum \upsilon_n$ converge to t. Then

(i) The series $\sum (u_n + \upsilon_n)$ converges to s + t

(ii) The series $\sum \lambda u_n$ converges to λs, where λ is a non zero real number.

Proof : (i) We have nth partial sum of $\sum (u_n + \upsilon_n)$

$= (u_1 + \upsilon_1) + (u_2 + \upsilon_2) + \ldots\ldots\ldots + (u_n + \upsilon_n)$

$= (u_1 + u_2 + \ldots\ldots + u_n) + (\upsilon_1 + \upsilon_2 + \ldots\ldots + \upsilon_n)$

$= S_n + T_n$

where $S_n = \sum_{r=1}^{n} u_r,\ T_n = \sum_{r=1}^{n} \upsilon_r$

Now $\lim_{n \to \infty} (S_n + T_n) = \lim_{n \to \infty} S_n + \lim_{n \to \infty} T_n$

$= s + t$

$\therefore$ The series $\sum (u_n + \upsilon_n)$ converges to s + t

(ii) We have

nth partial sum of $\sum \lambda u_n$

$= \lambda u_1 + \lambda u_2 + \ldots\ldots\ldots \lambda u_n$

$= \lambda (u_1 + u_2 + \ldots\ldots + u_n)$

$= \lambda S_n$

Now $\lim_{n \to \infty} \lambda S_n = \lambda s$

$\therefore$ The series $\sum \lambda u_n$ converges to λs

4.5 TWO SPECIAL SERIES

I. Geometric Series

Let us consider the geometric series

$1 + r + r^2 + \ldots\ldots\ldots\infty$, where r is the common ratio.

Case (i) First we consider the case where $|r| < 1$, i.e., $-1 < r < 1$.

Now $S_n = 1 + r + r^2 + \ldots\ldots\ldots + r^{n-1}$

$= \frac{1 - r^n}{1 - r} = \frac{1}{1 - r} - \frac{r^n}{1 - r}$

$\because |r| < 1,\ \lim_{n \to \infty} r^n = 0$

$\therefore \lim_{n \to \infty} S_n = \frac{1}{1 - r}$

The series is convergent, and it converges to $\frac{1}{1 - r}$.

Case (ii) Next we consider the case when $|r| \geq 1$

Then, there are four possibilities, viz, r = 1, r > 1, r = –1, r < –1

Let r = 1

We have $S_n = 1 + r + r^2 + \ldots\ldots\ldots + r^{n-1}$

$\because$ r = 1, S_n = 1 + 1 + 1 + ... + 1 (n terms)

= n

Clearly $S_n \to \infty$, as $n \to \infty$

$\therefore$ The series is divergent when r = 1.

Next, let r > 1.

We have $S_n = 1 + r + r^2 + \ldots\ldots\ldots + r^{n-1}$

$$= \frac{r^n - 1}{r - 1}$$

Now, $r^n \to \infty$ as $n \to \infty$ ($\because$ r > 1)

$\therefore S_n \to \infty$, as $n \to \infty$

$\therefore$ The series is divergent when r > 1

Next, let r = –1

Now $S_n \begin{cases} = 1, & \text{if } n \text{ is odd} \\ = 0, & \text{if } n \text{ is even} \end{cases}$

The sequence $\{S_n\}$ is divergent.

$\therefore$ The series is divergent when r = –1

Lastly, let r < –1

Clearly the sequence $\{S_n\}$ is divergent.

$\therefore$ The series is divergent when r < –1.

Thus, we see that the geometric series $1 + r + r^2 + \ldots\ldots\ldots\infty$ is convergent when | r | < 1, and divergent $|r| \geq 1$.

Note : The geometric series

$a + ar + ar^2 + \ldots\ldots\ldots\infty$ is convergent if | r | < 1. In this case, the sum of the series is $\frac{a}{1-r}$.

II. Harmonic Series

The series $\sum \frac{1}{n}$, i.e.

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots\ldots\infty$ is called Harmonic Series

Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots\ldots + \frac{1}{n}$

$H_1 = 1$

$H_2 = 1 + \frac{1}{2}$

$H_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$

$> 1 + 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2 \times \frac{1}{2}$

$H_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3 \times \frac{1}{2}$

Similarly $H_{16} > 1 + 4 \times \frac{1}{2}$

In this way $H_{2^n} > 1 + n \times \frac{1}{2}$

$\therefore H_n \to \infty$ as $n \to \infty$

$\therefore$ The series is divergent

Note 1. H_1, H_2, H_3, ... are named harmonic numbers by Americian computer scientist and mathematician Donald Ervin Knuth (born 1938).

2. We know that if a series $\sum u_n$ is convergent, then The converse $\lim_{n \to \infty} u_n = 0$. The converse of the result is not true, i.e., if $\lim_{n \to \infty} u_n = 0$, the series $\sum u_n$ may not converge.

For the harmonic series $\sum \frac{1}{n}$, $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0$.

But the series is not convergent.

Do You Know

"The name of the harmonic series derives from the concept of overtones or harmonics in music : The wavelengths of the overtones of a vibrating string are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ etc., of the string's fundamental

wavelength. Every term of the harmonic series after the first is the harmonic mean of the neighboring terms, so the terms form a harmonic progression; the phrases harmonic mean and harmonic progression likewise derive from music.''

—— Wikipedia

4.6 CAUCHY'S PRINCIPLE OF CONVERGENCE FOR SERIES

Theorem 4.3 The series $\sum_{n=1}^{\infty} u_n$ converges if an only if, given $\varepsilon > 0$, there exists a positive integer m such that $\left| \sum_{r=k+1}^{\infty} u_r \right| < \varepsilon$ where n, k > m (n>k)

Proof : Let $\{S_n\}$ be the sequence of partial sums of the series $\sum u_n$.

By Cauchy's Principle of convergence for sequence, $\{S_n\}$ is convergent if and only if given $\varepsilon > 0$, there exists a positive integer m such that

$|S_n - S_k| < \varepsilon$, for n, k > m (n > k)

Now $S_n - S_k$

$= (u_1 + u_2 + \ldots\ldots\ldots + u_k + u_{k+1} + u_{k+2} + \ldots\ldots + u_n)$

$= (u_1 + u_2 + \ldots\ldots\ldots + u_k)$

$= u_{k+1} + u_{k+2} + \ldots\ldots\ldots + u_n = \sum_{r=k+1}^{n} u_r$

The convergence of the sequence $\{S_n\}$ implies the convergence of $\sum u_n$.

$\therefore$ $\sum u_n$ is convergent if and only if given $\varepsilon > 0$, there exists a positive integer m such that

$$\left| \sum_{r=k+1}^{\infty} u_r \right| < \varepsilon, \text{ for } n, k > m, \quad (n>k)$$

4.7 SERIES OF POSITIVE TERMS

An infinite series $\sum u_n$, where $u_n > 0$ for all n, is called a series of positive terms or a positive term series.

Theorem 4.4 A services $\sum u_n$ of positive terms is convergent if and only if the sequence $\{S_n\}$ of partial sums is bounded above.

Proof : Let $S_n = u_1 + u_2 + u_3 + ...+ u_n$

Then $S_{n+1} - Sn = u_{n+1} > 0$, for all n.

$\therefore$ The sequence $\{S_n\}$ is a monotonic increasing sequence.

$\therefore$ $\{S_n\}$ is convergent if and only if it is bounded above.

$\therefore$ The series $\sum u_n$ is convergent if and only if the sequence $\{S_n\}$ is bounded above.

Note : $\sum u_n$ diverges to ∞, if $\{S_n\}$ is not bounded above.

Let us consider the harnomic series $\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ...$

We have seen that $H_{z^n} > 1 + n \times \frac{1}{2}$

This shows that $\{H_n\}$ is not bounded above.

$\therefore \sum \frac{1}{n}$ diverges to ∞.

Grouping of terms of a series

Let us consider the series

$1 - 1 + 1 - 1 + 1 - 1 + ... \infty$

clearly $u_n \begin{cases} = 1, & \text{if n is odd} \\ = -1, & \text{if is even} \end{cases}$

$S_n = u_1 + u_2 + u_3 + ...+ u_n$

$S_n \begin{cases} = 1, & \text{if n is odd} \\ = 0, & \text{if n is even} \end{cases}$

$\therefore$ $\{S_n\}$ is not convergent

$\therefore$ The series $1 - 1 + 1 - 1 + ... \infty$ is not convergent. But the series $(1 - 1) + (1 - 1) + ... \infty$

i.e. $0 + 0 + ... \infty$ is convergent.

Thus, we see that the grouping of terms has affected the nature of the series.

Now, let us consider the positive term series $u_1 + u_2 + u_3 + \ldots\ldots \infty$,

i.e. $\sum_{n=1}^{\infty} u_n$

Let the series $\sum_{n=1}^{\infty} u_n$ converge to S.

Let $S_n = u_1 + u_2 + u_3 + \ldots\ldots u_n$

clearly the sequence $\{S_n\}$ of partial sums is an increasing sequence of positive terms, bounded above.

Now ther terms of $\sum_{n=1}^{\infty} u_n$ are grouped. Let $\{T_n\}$ be the sequence associated with the new series.

clearly $\{T_n\}$ is a subsequence of $\{S_n\}$.

$\therefore \{T_n\}$ will also converge to S.

Thus, we see that a convergent series of positive terms remains convergent under grouping.

Similarly, a divergent series of positive terms will also remain divergent under grouping.

In short, grouping does not after the behaviour of a positive term series.

4.8 COMPARISON TEST FOR CONVERGENCE OF POSITIVE TERM SERIES :

There are some tests by which the convergence of positive term series can be examined without finding the sequence $\{S_n\}$ of partial sums. It may mentioned that it is not always easy to find the sequence $\{S_n\}$ of partial sums and check its convergence. In this unit, we will discuss one test, viz, the comparison test. Other tests will be discussed in the next unit.

Theorem 4.5 Let $\sum u_n$ and $\sum v_n$ be two series of positive terms.

I. Let there be a positive real number k (independent of n), and a positive integer m such that $u_n \leq k\, v_n, \ \forall n \geq m$. Then

(i) if $\sum v_n$ is convergent, $\sum u_n$ is convergent.

(ii) if $\sum u_n$ is divergent, $\sum \upsilon_n$ is divergent.

II. Suppose $\lim_{n \to \infty} \frac{u_n}{\upsilon_n} = L$, where L is a non zero real number. Then

$\sum u_n$ and $\sum \upsilon_n$ converge or diverge together.

III. Suppose there exists a positive integer m such that

$$\frac{u_n}{u_{n+1}} \geq \frac{\upsilon_n}{\upsilon_{n+1}}, \quad \forall\, n \geq m$$

Then, (i) if $\sum \upsilon_n$ is convergent, $\sum u_n$ is convergent.

(ii) if $\sum u_n$ is divergent, them $\sum \upsilon_n$ is divergent.

Proof : I Let $S_n = u_1 + u_2 + u_3 + \ldots\ldots + u_n$

$T_n = \upsilon_1 + \upsilon_2 + \upsilon_3 + \ldots\ldots \upsilon_n$

Then for all $n \geq m$

$S_n - S_m = u_{m+1} + u_{m+2} + \ldots\ldots + u_n$

$\leq k\,(\upsilon_{m+1} + \upsilon_{m+2} + \ldots\ldots \upsilon_n)$

$= k\,(T_n - T_m)$

or $S_n \leq k\,T_n + (S_m - k\,T_m)$

Putting $\lambda = S_m - k\,T_m$ we get

$$S_n \leq k\,T_n + \lambda \qquad (1)$$

(i) Now, if $\sum \upsilon_n$ is convergent, the sequence $\{T_n\}$ of partial sums is bounded above. Therefore, there exists a number β such that $T_n \leq B, \forall n$ (2)

From (1) and (2), we have

$S_n \leq k\ B + \lambda, \ \forall\ n \geq m$

$\therefore$ The sequence $\{S_n\}$ is bounded above

$\therefore \sum u_n$ is convergent.

(ii) Now, if $\sum u_n$ is divergent, the sequence $\{S_n\}$ of partial sums is not bounded above. Therefore, if A is any number, however large, there exists a positive integer p such that

$$S_n > A, \ \forall\ n \geq p$$

From (1) and (3),

$k\,T_n + \lambda \geq S_n > A, \ \forall\ n \geq \max\{m, m_1\}$

or $k T_n > A - \lambda$

i.e. $T_n > \frac{1}{k}(A - \lambda), \quad k \neq 0$

$\therefore$ The sequence $\{T_n\}$ is not bounded.

$\therefore \sum \upsilon_n$ is divergent.

II. We have $\lim_{n \to \infty} \frac{u_n}{\upsilon_n} = L$

clearly L > 0.

Let ε be a positive number such that $L - \varepsilon > 0$.

$\because \lim_{n \to \infty} \frac{u_n}{\upsilon_n} = L$, there exists a positive integer m such that

$$\left| \frac{u_n}{\upsilon_n} - 1 \right| < \varepsilon, \quad \forall\ n \geq m$$

or, $L - \varepsilon < \frac{u_n}{\upsilon_n} < L + \varepsilon, \quad \forall\ n \geq m$

or, $(L - \varepsilon)\,\upsilon_n < u_n < (L + \varepsilon)\upsilon_n, \quad \forall\ n \geq m$ (4)

From (4), we have $u_n < (L + \varepsilon)\upsilon_n, \quad \forall\ n \geq m$

$\therefore$ if υ_n is convergent, by I, is also convergent.

Again, from (4), we have $u_n > (L - \varepsilon)\upsilon_n, \quad \forall\ n \geq m$

$\therefore$ if $\sum \upsilon_n$ is divergent, by I, is also divergent.

Similarly, it can be proved that $\sum \upsilon_n$ converges of diverges with $\sum u_n$.

III. For $n \geq m$, we have

$$\frac{u_m}{u_n} = \frac{u_m}{u_{m+1}} \cdot \frac{u_{m+1}}{u_{m+2}} \cdots\cdots\cdots \frac{u_{n-1}}{u_n}$$

$$\geq \frac{\upsilon_m}{\upsilon_{m+1}} \cdot \frac{\upsilon_{m+1}}{\upsilon_{m+2}} \cdots\cdots\cdots \frac{\upsilon_{n-1}}{\upsilon_n}$$

$$= \frac{\upsilon_m}{\upsilon_n}$$

or $\frac{u_m}{u_n} \geq \frac{\upsilon_m}{\upsilon_n}$

or $u_m \upsilon_n \geq \upsilon_m\ u_n$

or $u_n \upsilon_m \leq u_m\ \upsilon_n$

$$\therefore\ u_n \le \frac{u_m}{\upsilon_m}\,\upsilon_n$$

Now $\frac{u_m}{\upsilon_m}$ is fixed as m is a fixed positive integer. We take $\frac{u_m}{\upsilon_m} = k$

$$\therefore u_n \le k\ \upsilon_n$$

$\therefore$ if $\sum \upsilon_n$ is convergent, by I, $\sum u_n$ is also convergent.

Also, if $\sum u_n$ is divergent, by I, $\sum \upsilon_n$ is divergent.

Remark : II is frequently used in working out problems.

4.9 p-SERIES (HYERHARMONIC SERIES)

Theorem 4.6 The series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+.....\infty$$

converges for p > 1, and diverges for p $\le$1.

Proof : Case I Let p > 1.

Let $\sum u_n = \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+.....\infty$

Here $u_n = \frac{1}{n^p}$

Now the series $\sum \upsilon_n$ is obtained from the series $\sum u_n$ by grouping the terms as follows.

$$1+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\left(\frac{1}{8^p}+.....+\frac{1}{15^p}\right)+.....\infty$$

Now $\upsilon_1 = 1$

$$\upsilon_2 = \frac{1}{2^p}+\frac{1}{3^p} < \frac{1}{2^p}+\frac{1}{2^p} = \frac{1}{2^{p-1}} = w_2$$

$$\upsilon_3 = \frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}$$

$$< \frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p} = \frac{4}{4^p} = \frac{1}{2^{2(p-1)}} = w_3$$

$$\upsilon_4 = \frac{1}{8^p}+\frac{1}{9^p}+.....+\frac{1}{15^p} < \frac{8}{8^p} = \frac{1}{2^{p\,(p-1)}} = w_4$$

.....

In this way, $w_n = \frac{1}{2^{(n-1)(p-1)}}$, $n \geq 2$

$\therefore \upsilon_n < w_n$, for all $n \geq 2$

Now $\sum w_n$ is geometric series with common ratio $\frac{1}{2^{p-1}}$.

$\because p > 1, \quad 0 < \frac{1}{2^{p-1}} < 1$

$\therefore \sum w_n$ is convergent.

$\therefore$ by Theorem 4.5 {I (i)}, $\sum \upsilon_n$ is convergent.

$\therefore \sum u_n$ is also convergent.

Case II Let p =1

The series is $1 + \frac{1}{2} + \frac{1}{3} + ...\infty$

We have already proved that the series is divergent.

Case III Let 0 < p < 1

$2^p < 2, \quad 3^p < 3, \quad 4^p < 4, ...$

i.e. $\frac{1}{2^p} > \frac{1}{2}, \quad \frac{1}{3^p} > \frac{1}{3}, \quad \frac{1}{4^p} > \frac{1}{4}$

i.e. $\frac{1}{n^p} > \frac{1}{n}$ i.e. $\frac{1}{n} < \frac{1}{n^p}$, for all $n \geq 2$

Now $\sum \frac{1}{n}$ is divergent

$\therefore \sum \frac{1}{n^p}$ is divergent (Theorem 4.5 {I (ii)}

Let $p \leq 0$

Then $\lim_{n \to \infty} \frac{1}{n^p} \neq 0$

$\therefore \sum \frac{1}{n^p}$ is not convergent.

Example 1 Examine the convergence of the series

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + ...\infty$$

Solution : Here $u_n = \dfrac{2n-1}{n(n+1)\,(n+2)}$

[1, 3, 5, ... is an arithemetic sequence.
$\therefore$ nth term = 1 + (n–1) 2 = 2n – 1.

Let us form the series $\sum \upsilon_n$, where $\upsilon_n = \dfrac{n}{n^3} = \dfrac{1}{n^2}$

[We note that the highest power in the numerator of u_n is n, and the highest power in the denominator is n^3. So we take $\upsilon_n = \dfrac{n}{n^3} = \dfrac{1}{n^2}$].

Now, we compare the two series $\sum u_n$ and $\sum \upsilon_n$.

$$\lim_{n\to\infty} \frac{u_n}{\upsilon_n} = \lim_{n\to\infty} \frac{2n-1}{n(n+1)(n+2)} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} \cdot \frac{n^2}{1}$$

$$= \lim_{n\to\infty} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 2,\ \text{a}$$

non zero real number.

$\sum \upsilon_n = \sum \dfrac{1}{n^2}$ is a p-series with p = 2.

$\sum \upsilon_n$ is convergent (Theorem 4.6)

So, by comparison test, $\sum u_n$ is convergent. (Theorem 4.5 II)



CHECK YOUR PROGRESS 1

Q.1: Test the convergence of the following series

(i) $\dfrac{1}{1.5} + \dfrac{1}{5.9} + \dfrac{1}{9.13} + \ldots\ldots\infty$

(ii) $\dfrac{1}{1.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} + \ldots\ldots\infty$

(iii) $\dfrac{1+2}{2^3} + \dfrac{1+2+3}{3^3} + \dfrac{1+2+3+4}{4^3} + \ldots\ldots\infty$

(iv) $\sum_{n=1}^{\infty} u_n$, where $u_n = \frac{\sqrt{n}}{n^2 - 1}$

(v) $\sum_{n=1}^{\infty} u_n$, where $u_n = \sqrt{n^4 + 1} - \sqrt{n^4 - 1}$

Q.2: Show that the series $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + ...\infty$ is convergent, and find its sum.

Q.3: If the series $\sum_{n=1}^{\infty} u_n$ of positive terms converges, show that $\sum_{n=1}^{\infty} u_n^2$ also converges.

Q.4: Examine the convergence of the series $\sum_{n=1}^{\infty} u_n$, where $u_n = \text{Sin}\frac{1}{n}$



4.10 LET US SUM UP

- If $\{u_n\}$ is a sequence of real numbers, then the expression $u_1 + u_2 + u_3 + ... \infty$ is called an infinite series of real numbers, or simply, infinite series or series. It is, generally, written as $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$

- Let us consider the series $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + ... \infty$
  If the sequence of partial sums $\{S_n\}$ is convergent, i.e., if there exists a number s such that $\lim_{n \to \infty} S_n = S$, then the series is called convergent, and s is called the sum of the series.

- If $\sum u_n$ is convergent, then $\lim_{n \to \infty} u_n = 0$.
  If $\lim_{n \to \infty} u_n \neq 0$, the series is not convergent.

- The geometric series $1 + r + r^2 + ...\infty$ is convergent if $| r | < 1$, i.e., if $-1 < r < 1$.

In this case, the sum of the series is $\frac{1}{1-r}$

- The harmonic series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....\infty$ is not convergent.
- The p-series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+.....\infty$ converges for p>1, and diverges for $p \leq 1$.



4.11 ANSWERS TO CHECK YOUR PROGRESS

Ans to Q No 1:

(i) $\frac{1}{1.5}+\frac{1}{5.9}+\frac{1}{9.13}+.....\infty$

For the arithmetic sequence, 1, 5, 9, ..., nth term = 1 + (n – 1) 4 = 4n –3

For the arithmetic sequence 5, 9, 13, ..., nth term = 5 + (n – 1) 4 = 4n +1

$$\therefore u_n = \frac{1}{(4n-3)\ (4n+3}$$

Let us form the series $\sum \upsilon_n$, where $\upsilon_n = \frac{1}{n^2}$

Now $\lim_{n \to \infty} \frac{u_n}{\upsilon_n} = \lim_{n \to \infty} \frac{1}{(4n-3)\ (4n+3)} \cdot \frac{n^2}{1}$

$$= \lim_{n \to \infty} \frac{1}{n^2(4-\frac{3}{n})\ (4+\frac{3}{n})} \cdot \frac{n^2}{1}$$

$$= \lim_{n \to \infty} \frac{1}{(4-\frac{3}{n})\ (4+\frac{3}{n})}$$

$= \frac{1}{16}$, a non zero real number.

Now $\sum \upsilon_n = \sum \frac{1}{n^2}$ is convergent.

$\therefore u_n$ is also convergent.

(ii) Try yourself. Convergent.

(iii) Here $u_n = \dfrac{1+2+3+.....+(n+1)}{(n+1)^3}$

$$= \frac{\frac{1}{2}(n+1)(n+2)}{(n+1)^3} = \frac{n+2}{2(n+1)^2}$$

Let us form the series $\sum v_n$, where $v_n = \dfrac{n}{n^2} = \dfrac{1}{n}$

Now $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{n+2}{2(n+1)^2}\cdot\dfrac{n}{1}$

$$= \lim_{n\to\infty} \frac{n\left(1+\frac{2}{n}\right)+2}{2n^2\left(1+\frac{1}{n}\right)^2}\cdot\frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}}{2\left(1+\frac{1}{n}\right)} = \frac{1}{2} \neq 0$$

Now, $\sum v_n = \sum \dfrac{1}{n}$ is divergent.

$\therefore \sum u_n$ is also divergent.

(iv) Here $u_n = \dfrac{\sqrt{n}}{n^2-1}$

Let us form the series $\sum v_n$, where $v_n = \dfrac{n^{\frac{1}{2}}}{n^2} = \dfrac{1}{n^{3/2}}$

$\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} \neq 0$

$\sum v_n = \sum \dfrac{1}{n^{3/2}}$ is convergent. $\left(p = \dfrac{3}{2} > 1\right)$

$\therefore \sum u_n$ is convergent.

(v) Here $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

Now, $u_n = \dfrac{\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right)\left(\sqrt{n^4+1} + \sqrt{n^4-1}\right)}{\sqrt{n^4+1} + \sqrt{n^4-1}}$

$= \dfrac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}}$

Let us form the series $\sum \upsilon_n$, where $\upsilon_n = \dfrac{1}{n^2}$

Now, $\lim\limits_{n \to \infty} \dfrac{u_n}{\upsilon_n} = \lim\limits_{n \to \infty} \dfrac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}} \cdot \dfrac{n^2}{1}$

$= \lim\limits_{n \to \infty} \dfrac{2}{n^2\left(\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}\right)} \cdot \dfrac{n^2}{1}$

$= \lim\limits_{n \to \infty} \dfrac{2}{\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}} = 1 \neq 0$

The series $\sum \upsilon_n = \sum \dfrac{1}{n^2}$ is convergent.

The series $\sum u_n$ is also convergent.

Ans to Q No 2: The series is $1 - \dfrac{1}{4} + \dfrac{1}{16} - \dfrac{1}{64} + \ldots\ldots\infty$

It is a geometric series with common ratio $r = -\dfrac{1}{4}$

$|r| = \left|-\dfrac{1}{4}\right| = \dfrac{1}{4} < 1$

$\therefore$ The series is convergent, and its sum = $\dfrac{1}{1-\left(-\frac{1}{4}\right)} = \dfrac{4}{5}$

Ans to Q No 3: It is given that the series of positive terms $\sum\limits_{n=1}^{\infty} u_n$ converges.

$\therefore \lim_{n \to \infty} u_n = 0$

$\therefore$ there is a positive integer m such that $u_n < 1$, for $n \geq m$.

$\therefore$ for $n \geq m$, $u_n^2 < u_n$

$\because \sum u_n$ is convergent,

$\therefore \sum u_n^2$ is also convergent

(Theorem 4.5 I (i))

Ans to Q No 4: Here the series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \text{Sin}\frac{1}{n}$

Let us form the series $\sum v_n$, where $v_n = \frac{1}{n}$

Now $\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{\text{Sin}\frac{1}{n}}{\frac{1}{n}}$

$= \lim_{\theta \to 0} \frac{\text{Sin}\theta}{\theta}$ (Putting $\frac{1}{n} = \theta$)

= 1, a non zero real number.

Now $\sum v_n$ is divergent.

$\therefore \sum u_n$ is also divergent



4.12 FURTHER READING

1) Fraleigh, John B : A first course in abstract Algebra, Pearson Education (Singapure) Pvt. Ltd. India Branch, Delhi - 11092.
2) Singh, S and Qazi, Z : Modern Algebra, Vikas Publicasting House Pvt. Ltd, New Delhi - 110014.
3) Macauley, M. : Section 6 : Fields and Galois Theory, http :// www.math.clemson.edu/~macaule/.
4) Talati, A. : Class Notes of Professor June Huh, 2022.
5) Singh, J. : Theory of Field Extension, Maharshi Dayanand University Press, Rohtal-124001.



4.13 MODEL QUESTIONS

Q 1: If the series $\sum_{n-1}^{\infty} u_n$ is convergent, prove that $\lim_{n \to \infty} u_n = 0$ What happens if $\lim_{n \to \infty} u_n \neq 0$. Show that the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent.

Q 2: Prove that the multiplication of each term of an infinite series by a non zero constant does not affect the convergence or divergence.

Q 3: Examine the convergence of the following series

(i) $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + ...\infty$

(ii) $1^2 + 2^2 + 3^2 + ...\infty$

(iii) $\sum u_n$, where $u_n = \frac{n+2}{n^3}$

(iv) $\sum u_n$, where $u_n = \frac{2n+1}{3n^3+1}$

(v) $\sum u_n$, where $u_n = \frac{4n^2 - n + 3}{n^3 + 2n}$

(vi) $\sum u_n$, where $u_n = \frac{\log_e n}{2n^3 - 1}$

[Hints : $\log_e n < n$, $\frac{1}{2n^3 - 1} \leq \frac{1}{n^3}$

$\therefore \frac{\log n}{2n^3 - 1} \leq \frac{n}{n^3} = \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ is convergent, $\therefore \sum u_n$ is also covergent]

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